

Holographic Thermalization, Stability of AdS, and the FPU Paradox

Venkat Balasubramanian¹, Alex Buchel^{1,2}, Stephen R. Green³, Luis Lehner² and Steven L. Liebling⁴

¹ *Department of Applied Mathematics,
University of Western Ontario,
London, Ontario N6A 5B7 Canada*

² *Perimeter Institute for Theoretical Physics,
Waterloo, Ontario N2L 2Y5, Canada*

³ *Department of Physics, University of Guelph
Guelph, Ontario N1G 2W1, Canada*

⁴ *Department of Physics, Long Island University
Brookville, NY 11548, U.S.A*

For a real massless scalar field in general relativity with a negative cosmological constant, we uncover a large class of spherically symmetric initial conditions that are close to AdS, but whose numerical evolution does not result in black hole formation. According to the AdS/CFT dictionary, these bulk solutions are dual to states of a strongly interacting boundary CFT that fail to thermalize at late times. Furthermore, as these states are not stationary, they define dynamical CFT configurations that do not equilibrate. We develop a two-timescale perturbative formalism that captures both direct and inverse cascades of energy and agrees with our fully nonlinear evolutions in the appropriate regime. We also show that this formalism admits a large class of quasi-periodic solutions. Finally, we demonstrate a striking parallel between the dynamics of AdS and the classic Fermi-Pasta-Ulam-Tsingou problem.

Introduction.—The gauge theory-string theory correspondence [1] has become a valuable tool to study nonequilibrium phenomena in strongly interacting QFTs [2–4]. In a particular limit, this correspondence links general relativity in $d+1$ dimensional asymptotically anti-de Sitter (AdS_{d+1}) spacetimes with conformal field theories in d dimensions. A question of particular importance in field theory is to understand the process of equilibration and thermalization. This corresponds, in the bulk, to the collapse of an initial perturbation to a black hole.

In the first detailed analysis [5] of dynamics of perturbations of global AdS_4 , Bizoń and Rostworowski argued that (except for special *nonresonant* initial data) the evolution of a real, massless, spherically symmetric scalar field *always* results in gravitational collapse, even for arbitrarily small initial field amplitude ϵ . At the linear level, this system is characterized by a normal mode spectrum with natural frequencies $\omega_j = 2j + 3$. Using weakly nonlinear perturbation theory, these authors described the onset of instability as a result of resonant interactions between the normal modes. Because of the presence of a vast number of resonances, they argued that this mechanism leads to a *direct turbulent cascade* of energy to high mode numbers, making gravitational collapse inevitable. Higher mode numbers are more sharply peaked, so this corresponds to an effect of gravitational focusing.

The analysis of [5] also showed that, for initial data consisting of a single mode, the dominant effect of resonant self-interaction could be absorbed into a constant shift in the frequency of the mode. (This time-periodic solution was later confirmed to persist at higher nonlinear order [6].) However, for two-mode initial data, additional resonances are present that cannot be absorbed into frequency shifts. The result is secular growth of

higher modes.

The turbulent cascade described in [5] is a beautiful mechanism for thermalization of strongly coupled QFTs with holographic gravitational duals. However, it was recently pointed out that this cascade argument breaks down if *all* modes are initially populated, and the mode amplitudes fall off sufficiently rapidly for high mode numbers [7]. In this case, the resonant effects may once again be absorbed into frequency shifts and black hole collapse is avoided. Low-lying modes have broadly distributed bulk profiles. Thus, one might expect that if the initial scalar profile is broadly distributed, its evolution might not result in gravitational collapse (see also [8–10]). This prediction was verified numerically [11]. The physical mechanism responsible for collapse/non-collapse of small amplitude initial data is a competition between two effects: gravitational focusing and nonlinear dispersion of the propagating scalar field. If the former dominates, gravitational collapse ensues [5]. If the latter does, the system evolves without approaching any identifiable static or stationary solution—the perturbed boundary CFT neither thermalizes nor equilibrates at late times [11].

The perturbation theory employed in [5] cannot make predictions at late times. (The growth of secular terms in the expansion leads to a breakdown at time $t \propto 1/\epsilon^2$.) It also does not properly take into account energy transfer between modes. In this Letter, we undertake a thorough analysis of the dynamics of AdS by making use of a new perturbative formalism for analyzing the effect of resonances on the evolution of this system *that is valid for long times*. We also perform fully nonlinear GR simulations (see [7, 11] for details of our numerical implementation). In the process we uncover a close relationship be-

tween the dynamics of AdS and the famous Fermi-Pasta-Ulam-Tsingou (FPU) problem [12]. Our formalism is based on a two-timescale approach [13], where we introduce a new “slow time” $\tau = \epsilon^2 t$. The timescale τ characterizes energy transfers between modes, whereas the “fast time” t characterizes the original normal modes. Importantly, this formalism allows one to study the system for long times and examine energy transfer between modes. In the following we describe the *Two Time Framework* (TTF) and determine a large class of quasi-periodic solutions that extends the single-mode periodic solutions of [5, 6]. These solutions have finely tuned energy spectra such that the net energy flow into each mode vanishes, and they appear to be stable to small perturbations within both TTF and full numerical simulations. We then study the behavior of two-mode initial data of [5] under both approaches. Finally, we use the TTF equations to draw an interesting parallel between scalar collapse in AdS and the FPU problem of thermalization of nonlinearly coupled oscillators [12].

Model.—Following [5], we consider a self-gravitating, real scalar field ϕ in asymptotically AdS₄ spacetime. Imposing spherical symmetry, the metric takes the form

$$ds^2 = \frac{1}{\cos^2 x} (-Ae^{-2\delta} dt^2 + A^{-1} dx^2 + \sin^2 x d\Omega^2), \quad (1)$$

where we set the asymptotic AdS radius to one. Spherical symmetry implies that A , δ and ϕ are functions of time $t \in (-\infty, \infty)$ and the radial coordinate $x \in [0, \frac{\pi}{2})$.

In terms of the variables $\Pi \equiv e^\delta \dot{\phi}/A$ and $\Phi \equiv \phi'$, the equation of motion for ϕ is

$$\ddot{\phi} = \left(\dot{A}e^{-\delta} - A\dot{\delta}e^{-\delta} \right) \Pi + A^2 e^{-2\delta} \Phi' + \left(\frac{2}{\sin x \cos x} A^2 e^{-2\delta} + AA'e^{-2\delta} - A^2 e^{-2\delta} \delta' \right) \Phi, \quad (2)$$

while the Einstein equation reduces to the constraints,

$$A' = \frac{1 + 2 \sin^2 x}{\sin x \cos x} (1 - A) + \sin x \cos x A (|\Phi|^2 + |\Pi|^2), \quad (3)$$

$$\delta' = -\sin x \cos x (|\Phi|^2 + |\Pi|^2). \quad (4)$$

Two Time Framework.—TTF consists of defining the slow time $\tau = \epsilon^2 t$ and expanding the fields as

$$\phi = \epsilon \phi_{(1)}(t, \tau, x) + \epsilon^3 \phi_{(3)}(t, \tau, x) + O(\epsilon^5), \quad (5)$$

$$A = 1 + \epsilon^2 A_{(2)}(t, \tau, x) + O(\epsilon^4), \quad (6)$$

$$\delta = \epsilon^2 \delta_{(2)}(t, \tau, x) + O(\epsilon^4). \quad (7)$$

It is possible to go beyond $O(\epsilon^3)$ through the introduction of additional slow time variables but such an analysis is beyond the scope of this Letter. The order of approximation used here is sufficient to capture the key aspects of weakly nonlinear AdS collapse in the $\epsilon \rightarrow 0$ limit.

Perturbative equations are derived by substituting the expansions (5)–(7) into the equations of motion (2)–(4), and equating powers of ϵ . It is important to note that, when taking time derivatives of a function of both time variables we have $d/dt = \partial_t + \epsilon^2 \partial_\tau$. At $O(\epsilon)$, we obtain the wave equation for $\phi_{(1)}$ linearized off exact AdS,

$$\frac{\partial^2 \phi_{(1)}}{\partial t^2} = \phi_{(1)}'' + \frac{2}{\sin x \cos x} \phi_{(1)}' \equiv -L \phi_{(1)}. \quad (8)$$

The operator L has eigenvalues $\omega_j^2 = (2j+3)^2$ ($j = 0, 1, 2, \dots$) and eigenvectors $e_j(x)$ (“oscillons”) [5]. Explicitly,

$$e_j(x) = d_j \cos^3 x {}_2F_1 \left(-j, 3+j; \frac{3}{2}; \sin^2 x \right), \quad (9)$$

with $d_j = 4\sqrt{(j+1)(j+2)}/\sqrt{\pi}$. The oscillons form an orthonormal basis under the inner product

$$(f, g) = \int_0^{\pi/2} f(x)g(x) \tan^2 x dx. \quad (10)$$

The general real solution to (8) is then

$$\phi_{(1)}(t, \tau, x) = \sum_{j=0}^{\infty} (A_j(\tau) e^{-i\omega_j t} + \bar{A}_j(\tau) e^{i\omega_j t}) e_j(x), \quad (11)$$

where $A_j(\tau)$ are arbitrary functions of τ , to be determined later.

At $O(\epsilon^2)$ the constraints (3)–(4) may be solved to give

$$A_{(2)}(x) = -\frac{\cos^3 x}{\sin x} \int_0^x (|\Phi_{(1)}(y)|^2 + |\Pi_{(1)}(y)|^2) \tan^2 y dy, \quad (12)$$

$$\delta_{(2)}(x) = -\int_0^x (|\Phi_{(1)}(y)|^2 + |\Pi_{(1)}(y)|^2) \sin y \cos y dy. \quad (13)$$

Finally, at $O(\epsilon^3)$ we obtain the equation for $\phi_{(3)}$,

$$\frac{\partial^2 \phi_{(3)}}{\partial t^2} + L \phi_{(3)} + 2 \frac{\partial^2 \phi_{(1)}}{\partial t \partial \tau} = S_{(3)}(t, \tau, x), \quad (14)$$

where the source term is

$$S_{(3)} = [\partial_t(A_{(2)} - \delta_{(2)}) - 2(A_{(2)} - \delta_{(2)})L] \phi_{(1)} + (A_{(2)}' - \delta_{(2)}') \phi_{(1)}'. \quad (15)$$

The solutions (12)–(13) for $A_{(2)}$ and $\delta_{(2)}$ are to be substituted directly into $S_{(3)}$. In general, the source term $S_{(3)}$ will contain resonant terms (*i.e.*, terms proportional to $e^{\pm i\omega_j t}$). As noted in [5], for all triads (j_1, j_2, j_3) , resonances occur at

$$\omega_j = \omega_{j_1} + \omega_{j_2} - \omega_{j_3}. \quad (16)$$

In ordinary perturbation theory these resonances lead to secular growths in $\phi_{(3)}$. However, it was noted in [5]

that in some cases the growths may be absorbed into frequency shifts. TTF provides a natural way to handle these resonances by taking advantage of the new term $2\partial^2\phi_{(1)}/\partial t\partial\tau$ in (14) and the freedom in $A_j(\tau)$.

We now project (14) onto an individual oscillon mode e_j and substitute our solution for $\phi_{(1)}$,

$$\left(e_j, \frac{\partial^2\phi_{(3)}}{\partial t^2} + \omega_j^2\phi_{(3)}\right) - 2i\omega_j \left(\frac{dA_j}{d\tau}e^{-i\omega_j t} - \frac{d\bar{A}_j}{d\tau}e^{i\omega_j t}\right) = (e_j, S_{(3)}) . \quad (17)$$

By exploiting the presence of terms proportional to $e^{\pm i\omega_j t}$ on the left hand side of the equation, we may cancel off the resonant terms on the right hand side. Denoting by $f[\omega_j]$ the part of f proportional to $e^{i\omega_j t}$, we set

$$-2i\omega_j \frac{dA_j}{d\tau} = (e_j, S(t, \tau, x))[-\omega_i] = \sum_{klm} \mathcal{S}_{klm}^{(j)} \bar{A}_k A_l A_m, \quad (18)$$

where $\mathcal{S}_{klm}^{(j)}$ are real constants representing different resonance channel contributions. The right hand side is a cubic polynomial in A_j and \bar{A}_j . Thus, we have obtained a set of coupled first order ODEs in τ for A_j , which we shall refer to as the *TTF equations*. The equations are to be solved given the initial conditions for ϕ . This procedure fixes the arbitrariness in the solution (11) for $\phi_{(1)}$. While we could continue and solve for $\phi_{(3)}$, this would be of little interest since the lack of resonances remaining in (14) means that $\phi_{(3)}$ will remain bounded.

Under evolution via the TTF equations, both the amplitude and phase of the complex coefficients $A_j(\tau)$ can vary. Thus, in contrast to the perturbative analysis in [5], the energy per mode $E_j = \omega_j^2 |A_j|^2$ can change with time in a very nontrivial manner. However, it can be checked that the total energy $E = \sum_j E_j$ is conserved. TTF thus describes an energy-conserving, dynamical system. We also remark that the TTF equations possess a scaling symmetry $A_j(\tau) \rightarrow \epsilon A_j(\tau/\epsilon^2)$. This symmetry was observed in Fig. 2b of [5], which indicates that the instability mechanism is captured by TTF.

In practice, it is necessary to truncate the TTF equations at finite $j = j_{\max}$. We evaluated the coefficients $\mathcal{S}_{klm}^{(j)}$ up to $j_{\max} = 47$. In particular, under truncation to $j_{\max} = 0$, the equations reduce to

$$i\pi \frac{dA_0}{d\tau} = 153A_0^2\bar{A}_0, \quad (19)$$

with solution $A_0(\tau) = A_0(0) \exp(-i\frac{153}{\pi}|A_0(0)|^2\tau)$. This reproduces precisely the single-mode frequency shift result of [5].

Quasi-periodic solutions.—To understand the dynamics of TTF, we first look for quasi-periodic solutions. For $j_{\max} = 0$ this is the periodic solution above. For general $j_{\max} > 0$ we take as ansatz $A_j = \alpha_j \exp(-i\beta_j\tau)$,

where $\alpha_j, \beta_j \in \mathbb{R}$ are independent of τ . These solutions have $E_j = \text{constant}$, so they represent a balancing of energy fluxes such that each mode has constant energy. Substituting into the TTF equations, we find that the τ -dependence can be canceled by imposing the relation

$$\beta_j = \beta_0 + j(\beta_1 - \beta_0). \quad (20)$$

This leaves $j_{\max} + 1$ algebraic equations,

$$-2\omega_j \alpha_j [\beta_0 + j(\beta_1 - \beta_0)] = \sum_{kmn} \mathcal{S}_{kmn}^{(j)} \alpha_k \alpha_m \alpha_n, \quad (21)$$

for $j_{\max} + 3$ unknowns $(\beta_0, \beta_1, \{\alpha_j\})$. The equations for $j = 0, 1$ may be used to eliminate (β_0, β_1) , leaving $j_{\max} - 1$ equations to be solved for $\{\alpha_j\}$ —two parameters of underdetermination. The scaling symmetry allows for elimination of one of the parameters, so we set $\alpha_{j_r} = 1$ for some fixed $0 \leq j_r < j_{\max}$. Taking the remaining free parameter to be α_{j_r+1} and requiring solutions to be insensitive to the value of j_{\max} (i.e., stable to truncation), it is straightforward to construct solutions perturbatively in $\alpha_{j_r+1}/\alpha_{j_r}$. We find a single solution for $j_r = 0$ and precisely two otherwise (see Fig. 1).

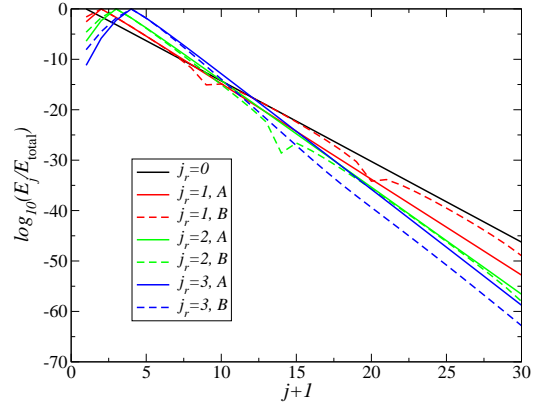


FIG. 1. (Color online) Energy spectra of quasi-periodic solutions with $\alpha_{j_r+1}/\alpha_{j_r} = 0.1$ for $j_r = 0, 1, 2, 3$. Dashed and solid lines distinguish different branches for $j_r > 0$. The solid branch is well-approximated by an exponential to each side of j_r . For $\alpha_{j_r+1}/\alpha_{j_r}$ too large, it becomes difficult to obtain solutions to (21), but for $j_r = 0$, we can go up to $\alpha_1/\alpha_0 \approx 0.42$. (Constructed for $j_{\max} = 30$.)

Stability of quasi-periodic solutions.—Ref. [6] extended single-mode, time-periodic solutions to higher order in ϵ and found these solutions to be stable to perturbations. Similarly, we examine the stability of our extended class of quasi-periodic solutions, both using full numerical relativity simulations (implementation described in [7]) and by numerically solving the TTF ODEs.

We consider initial data $A_j(0) = \epsilon \exp(-\mu j)/(2j+3)$, which well-approximates quasi-periodic solutions with $j_r = 0$. Varying μ and also adding random perturbations,

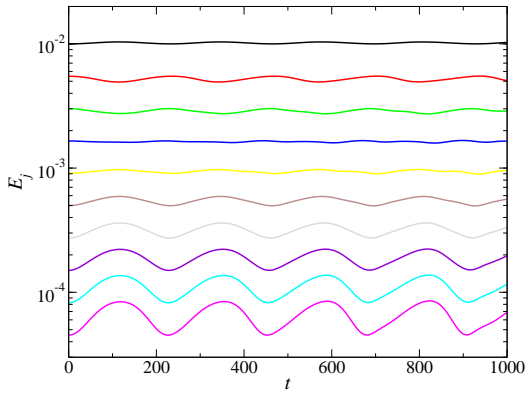


FIG. 2. (Color online) Energy per mode for $0 \leq j \leq 9$ for the TTF solution with initial data $A_j(0) \propto \exp(-0.3j)/(2j+3)$.

we observe periodic oscillations about the quasi-periodic solution, providing evidence for stability (see Fig. 2). For smaller values of μ , energy levels are more closely spaced, resulting in more rapid energy transfers between modes in Eq. (18), leading to larger amplitude oscillations. Likewise, larger random perturbations increase the amplitude of oscillation, as our initial data deviates more strongly from a quasi-periodic solution. Results from TTF and full numerical relativity simulations are in close agreement.

Two-mode initial data.—Our main interest is to understand which initial conditions can be expected to collapse. Thus it is necessary to study initial data that are *not* expected to closely approximate a quasi-periodic solution. A particularly interesting case consists of two modes initially excited (all others zero) as this case was key to the argument of [5] showing the onset of the turbulent cascade. In contrast to results of the previous section, two-mode initial data

$$A_j(0) = \epsilon (\delta_j^0 + \kappa \delta_j^1) \quad (22)$$

involves considerable energy transfer among modes provided κ is sufficiently large. [For $\kappa \ll 1$, (22) may be considered as a perturbation about single-mode data.] We looked at several choices of κ using both TTF and full numerical relativity, with similar results. Here we restrict to $\kappa = 3/5$ —the equal-energy case.

The upper envelope of $\Pi^2(x=0)$ has frequently been used as an indicator of the onset of instability [5, 7, 11]. We plot this quantity in Fig. 3, both for full GR simulations and TTF solutions with varying j_{\max} . In the full GR simulation, $\Pi^2(x=0)$ grows initially, but, in contrast to blowup observed in [5] for a Gaussian scalar field profile, it then decreases close to its initial value. This *recurrence* phenomenon repeats and collapse never occurs for as long as we have run the simulation. Recurrence was also observed in previous work [11] for broadly distributed Gaussian profiles.

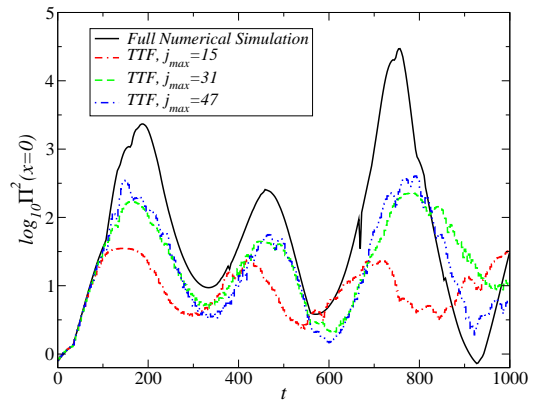


FIG. 3. (Color online) Full numerical and TTF results for 2-mode equal-energy initial data with $\epsilon = 0.109$. Notice that as j_{\max} is increased, the TTF solutions achieve better agreement with the full numerics. The recurrence behavior observed in the full numerical solution is reasonably well captured by TTF.

Also in Fig. 3, TTF solutions appear to converge to the full numerical GR solution as j_{\max} is increased. (Strictly speaking, the TTF and numerical approaches converge as both $j_{\max} \rightarrow \infty$ and $\epsilon \rightarrow 0$.) This convergence illustrates nicely the cascade/collapse mechanism: Higher- j modes are more sharply peaked at $x=0$, so as the (conserved) energy is transferred to these modes, $\Pi^2(x=0)$ attains higher values. Truncating the system at finite j_{\max} artificially places a bound on values of $\Pi^2(x=0)$ that can be reached. In particular, $\Pi^2(x=0)$ can never blow up for $j_{\max} < \infty$.

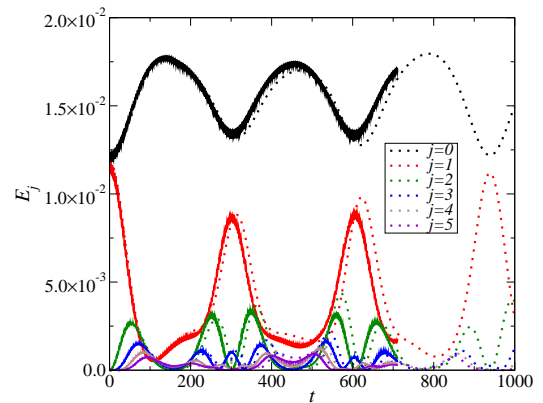


FIG. 4. (Color online) Full numerical (solid) and TTF (dotted) energy per mode for 2-mode equal-energy initial data. Notice the repeated approximate return of the initial energies to the first two modes. (The horizontal offset is partially attributed to a slight difference in time-normalization for our numerical and TTF codes.) Here, $j_{\max} = 47$.

It is useful to examine the solution mode by mode, and in Fig. 4 we show the energy per mode as a function

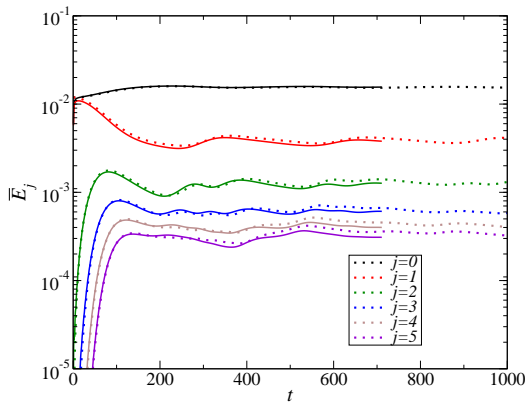


FIG. 5. (Color online) Running time-average energy per mode for equal-energy two-mode initial data. Solid lines from full numerical simulation, dotted from TTF.

of time. Initially, energy is distributed evenly between modes $j = 0, 1$. It then flows out of $j = 1$ to mode $j = 2$, then $j = 3$, etc. At some point in time, energy begins to flow *back* to mode $j = 1$. By $t = 300$ the state has nearly returned to the original configuration. This recurrence behavior then repeats. Fig. 5 plots the running time-average energy per mode $\bar{E}_j(t) \equiv t^{-1} \int_0^t E_j(t') dt'$. Rather than cascading to ever-higher modes, the energy sloshes primarily between low- j modes, in a sort of “metastable” state. We never observe thermalization, *i.e.*, no equipartition of energy occurs.

Figs. 4 and 5 are remarkably similar in appearance to those of Fermi, Pasta, Ulam, and Tsingou [12]. (Compare with Figs. 4.1 and 4.2 of [14]). FPU numerically simulated the behavior of a collection of nonlinearly coupled harmonic oscillators and expected to see thermalization. Instead, they observed the same sort of recurrence we see here. Indeed, as the TTF formulation (18) of our system makes clear, small-amplitude scalar collapse in AdS reduces precisely to a (infinite) set of nonlinearly coupled oscillators, so the similar behavior should not be surprising. More precisely, our system is related to the FPU β -model [14]. (Of course, the particular resonances and nonlinear interactions differ between our system and FPU.) Understanding when the FPU system of oscillators can be expected to thermalize is a longstanding problem in nonlinear dynamics, and is indeed known as the FPU *paradox* (see *e.g.*, [14–16]).

Discussion.—Common intuition suggests that a finite-sized strongly interacting system, driven off-equilibrium, even by a small amount, eventually thermalizes. This thermalization would imply, via AdS/CFT, that arbitrarily small perturbations about global AdS *must* result in gravitational collapse at late times. However, we have uncovered in this Letter a large class of initial conditions for a massless, self-gravitating real scalar field in AdS₄, that *fail* to collapse. We constructed and evolved these

initial conditions within a newly proposed TTF, as well as through full numerical GR simulations. TTF shows that scalar perturbations of AdS are in the same universality class as the famous FPU problem [12]. Thus, perturbed AdS spacetimes act as a holographic bridge between non-equilibrium dynamics of CFTs and the dynamics of nonlinearly coupled oscillators and the FPU paradox. In this Letter we focused on dynamics of low-energy 2+1 dimensional CFT excitations “prepared” with nonzero expectation values of dimension three (marginal) operators. Extensions to higher-dimensional CFTs, as well as to states generated by (ir)relevant operators are straightforward.

Acknowledgments: We would like to thank A. Polkovnikov and J. Santos for interesting discussions and correspondence. We also thank D. Minic and A. Zimmerman for pointing us to the FPU problem. This work was supported by the NSF under grants PHY-0969827 & PHY-1308621 (LIU), NASA under grant NNX13AH01G, NSERC through a Discovery Grant (to A.B. and L.L.) and CIFAR (to L.L.). S.R.G. acknowledges support by a CITA National Fellowship. L.L. and S.L.L. also acknowledge the Centre for Theoretical Cosmology at the University of Cambridge for their hospitality and its participants for helpful discussions at the “New Frontiers in Dynamical Gravity” meeting during which this work was completed. Research at Perimeter Institute is supported through Industry Canada and by the Province of Ontario through the Ministry of Research & Innovation. Computations were performed at Sharcnet.

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